臺北市立大學

107 學年度第一學期學士班二、三年級轉學生招生考試試題

系　　別：數學系（三年級）
科　　目：高等微積分
考試時間：90 分鐘【8:30−10:00】
總　　分：100 分

不得使用計算機或任何儀具。

※　**注意**：不必抄題，作答時請將試題題號及答案依照順序寫在答卷上；**限用藍色或黑色筆作答**，使用其他顏色或鉛筆作答者，所考科目以零分計算。**(於本試題紙上作答者，不予計分。)**

計算證明題（每題 10 分，共 100 分）

1. Suppose that a_n is a sequence in $\mathbb{R}$, and $a_n > 0$ for all $n \in \mathbb{N}$.

   Prove that $\limsup \sqrt[n]{a_n} \le \limsup \frac{a_{n+1}}{a_n}$.

2. Let a_n be a Cauchy sequence in $\mathbb{R}$. Prove that a_n is a convergent sequence in $\mathbb{R}$.

3. Let $[a,b]$ be a closed, bounded interval and $f_n \to f$ uniformly on $[a,b]$ as $n \to \infty$. Suppose that each f_n is integrable on $[a,b]$. Prove that f is integrable on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx .$$

4. Suppose that E is a compact set in $\mathbb{R}^n$. Show that E is closed and bounded in $\mathbb{R}^n$.

5. Suppose that E is a nonempty compact subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}^n$ is a continuous function. Show that f is uniformly continuous on E.

6. Suppose that $f : \mathbb{N} \to \mathbb{R}$. If $\lim_{n\to\infty} f(n+1) - f(n) = L$. Prove that $\lim_{n\to\infty} \frac{f(n)}{n} = L$ exists.

7. Suppose that f is differentiable at every point in a closed, bounded interval $[a,b]$. Prove that if $\frac{d}{dx}f$ is increasing on (a,b), then $\frac{d}{dx}f$ is continuous on (a,b).

8. Let f be defined on $[0,1]$ as

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ with } \gcd(m,n) = 1, \end{cases}$$

where $m, n \in \mathbb{N}$. Prove that f is continuous only at every irrational point in $[0,1]$

9. Find the maximum and minimum values of $f(x,y,z) = xyz$ subject to $x = y$ and $x^2 + z^2 = 1$

10. Compute the second-order Taylor formula for $f(x,y) = e^x \sin y$ around $(0, \pi/6)$.